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Fuel- or Time-Optimal Transfers Between Coplanar, Coaxial Ellipses Using Lambert's Theorem

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Undoubtedly, minimum-fuel and minimum-time orbit transfer are the two major goals of the optimal orbit maneuver. This paper considers two coplanar elliptic orbits when the apsidal lines coincide. We analytically find the conditions for the two-impulse minimum-time transfer orbit using Lambert's theorem. In the minimum-time transfer the transfer time is a decreasing function of a variable related to the transfer orbit's semimajor axis. Consequently there exists no unique minimum-time solution. Thus for the minimum-time case, there is a limiting solution only; however, there exists a unique solution in the case of minimum-fuel transfer, for which we find the necessary and sufficient conditions. Furthermore, as a special case, we consider when the transfer angle is 180 deg. In this case we show that we obtain the fuel-optimal Hohmann transfer orbit. We also derive the Hohmann transfer time and delta-velocity equations from more general equations, which are also obtained using Lambert's theorem. There is a tradeoff between minimum-time and minimum-fuel transfer. Finally, we propose an optimal coplanar orbit maneuver algorithm for trading off the minimum-time goal against the minimum-fuel goal.

Introduction

TRAJECTORY optimization with respect to fuel was performed as early as 1963 by Lawden[1] using the primer vector, but the optimal-time transfer was not addressed. The time vs fuel tradeoff for the rendezvous problem was addressed by Prussing and Chiu,[2] who obtained the minimum-fuel, multiple-impulse, time-fixed solutions for coplanar circular rendezvous problems. They also showed that the Hohmann transfer is the time-open solution for the optimal rendezvous. Prussing and Chiu[2] developed an iterative minimization procedure to determine the optimal number of impulses and their positions and time using Lawden's primer vector. The Hohmann transfer is the minimum-fuel two-impulse transfer between two coplanar circular orbits.[3] For trajectory optimization problem see Bryson and Ho[4] and the references cited therein. Betts[5] gave an excellent general survey of numerical methods for trajectory optimization.

A typical application of Lambert's theorem is to determine the transfer orbit from the connecting two position vectors and the transfer time. See Refs. 6 and 7 and the references cited within. It is possible to perform, however, the optimal-fuel or optimal-time transfer using Lambert's theorem. In this paper we formulate the transfer time and delta-velocity equations using Lambert's theorem. Then we derive the minimum-time transfer condition for the two-impulse elliptic-to-elliptic transfer. The necessary and sufficient conditions are also derived for minimum-fuel transfer when the transfer angle is anywhere between zero and 360 deg. Furthermore, we show that the Hohmann transfer is the minimum-fuel two-impulse solution when the transfer angle is 180 deg.

This optimal transfer is being developed with satellite orbit maneuvers, especially satellite altitude maneuvers, in mind. More than two impulsive maneuvers are not being considered because when the initial orbit radius is 11.94 times smaller than the target orbit radius, the absolute minimum-fuel circle-to-circle coplanar transfer is the Hohmann transfer; the orbits we are mainly interested in are all smaller than that.[8,9] Thus, we consider just the two-impulse solutions. On the other hand, the optimal transfer time is another major factor in our development because it is important to know the time a spacecraft takes to reach a particular point in the orbit. In this case we find the minimum-time solution assuming that two impulses are used. This is necessary in the case of spacecraft interception and rendezvous. We consider the theories and algorithms involved in the fuel vs time tradeoff.

In this paper we find the two-impulse minimum-time transfer orbit analytically. Moreover, we derive the necessary and sufficient conditions for the minimum-fuel orbit transfer problem using Lambert's theorem. We also state an algorithm to find the tradeoff between minimum-fuel and minimum-time orbit transfer.

Problem Formulations

We assume two elliptic coplanar orbits when their apsidal lines coincide. Figure 1 shows the initial orbit radius r_1, target orbit radius r_2, the initial orbit semimajor axis a_1, the target orbit semimajor axis a_2, the difference in the true anomaly (transfer angle) Δf, and the cord length c. The semiperimeter of the space triangle is given by

$$s = \frac{r_1 + r_2 + c}{2} \tag{1}$$

Following Battin's formulation,[10] define a constant λ by

$$\lambda^2 = (s - c)/s \tag{2}$$

Note that $\lambda \in [-1, 1]$, and it is closely related to the transfer angle Δf. Thus, λ relates to the cord length. For example, $\lambda = 0$ implies

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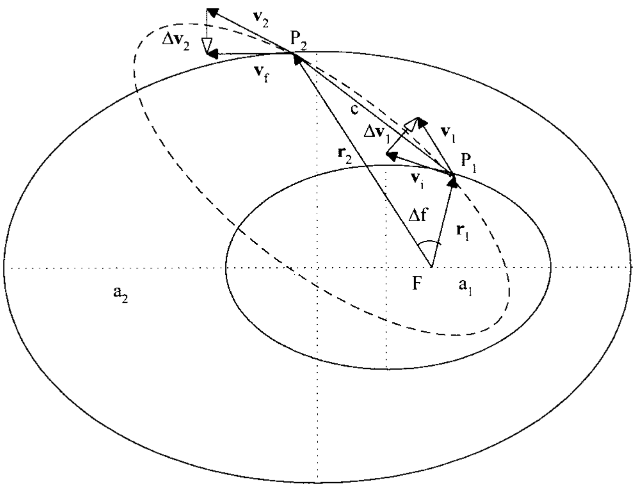

Fig. 1 Coplanar orbit transfer diagram.

$\Delta f = \pi$, and $\lambda = \pm 1$ implies $\Delta f = 0$. Also, the variable x is defined by

$$x^2 = 1 - a_m/a \tag{3}$$

where a_m is the semimajor axis of the minimum-energy elliptic transfer orbit given as

$$a_m = s/2 \tag{4}$$

and a is the semimajor axis of the transfer orbit.

Transfer-Time Equation

Using Lambert's theorem, we have the transfer-time equation for elliptic (hyperbolic) transfer orbits as[10]

$$\Delta t = \sqrt{\frac{a_m^3}{\mu}}\left[\frac{\alpha - \sin\alpha}{\sin^3(\alpha/2)} - \lambda^3 \frac{\beta - \sin\beta}{\sin^3(\beta/2)}\right] \tag{5}$$

where $\mu = 3.986 \times 10^5$ km^3/s^2 is the gravitational constant of the Earth. For hyperbolic orbits we replace sin with sinh and multiply Eq. (5) by -1. Following Battin,[10] we define

$$\alpha = 2\arccos(x), \qquad (\text{hyperbolic; } x = \cosh\alpha/2) \tag{6}$$

$$\beta = 2\arccos(y), \qquad (\text{hyperbolic; } y = \cosh\beta/2) \tag{7}$$

and

$$y = \sqrt{1 - \lambda^2(1 - x^2)} \tag{8}$$

Thus we have expressed the transfer-time equation in terms of the new variable x. Equation (5) can be utilized to find the minimum-time transfer orbit. In the next section we differentiate Δt with respect to x to find the minimum transfer-time orbit.

Before we consider this derivative, however, it is helpful to express α and β in terms of geometric parameters s, c, and a. When Eq. (4) is substituted into Eq. (3), we obtain

$$x = \pm\sqrt{1 - s/2a} \tag{9}$$

Equating the preceding equation with Eq. (6) and using a simple trigonometric identity, we obtain

$$\sin^2(\alpha/2) = s/2a \tag{10}$$

Similarly, substitute Eqs. (2) and (9) into Eq. (8) to obtain $y = \sqrt{[1 - (s - c)/2a]}$. Also from the definition (7) and a simple trigonometric identity, we obtain

$$\sin^2(\beta/2) = (s - c)/2a \tag{11}$$

Finally, we have α and β in terms of s, c, and a in Eqs. (10) and (11).

Delta-Velocity Equation

The relation between the transfer orbit's semimajor axis and the fuel usage can be found from the total delta-velocity vector, using Lambert's theorem.[10] In this section the total delta-velocity vector shall be written as a function of the variable x to perform fuel optimization in the sequel.

Let $\boldsymbol{v}_1$ be the transfer orbit velocity at P_1 and $\boldsymbol{v}_i$ be the initial orbit velocity at P_1 (Fig. 1). Then the delta velocity is defined as

$$\Delta\boldsymbol{v}_1 = \boldsymbol{v}_1 - \boldsymbol{v}_i \tag{12}$$

$$\boldsymbol{v}_1 = \frac{1}{\eta}\sqrt{\frac{\mu}{a_m}}\left\{\left[2\lambda\frac{a_m}{r_1} - (\lambda + x\eta)\right]\boldsymbol{i}_{r1} + \left(\sqrt{\frac{r_2}{r_1}}\sin\frac{\Delta f}{2}\right)\boldsymbol{i}_n \times \boldsymbol{i}_{r1}\right\} \tag{13}$$

where $\boldsymbol{i}_{r1}$ is a unit vector defining the direction of P_1 from the force center and $\boldsymbol{i}_n$ is a unit vector normal to the orbital plane. Define η by

$$\eta^2 = (1 - \lambda^2) + 4\lambda\sin^2(\psi/2) \tag{14}$$

for an elliptic orbit. Moreover, ψ is given as

$$\psi = (\alpha - \beta)/2 \tag{15}$$

where α and β are given in Eqs. (10) and (11). Also, because we assumed two elliptic coplanar orbits, the initial orbit velocity is given by

$$\boldsymbol{v}_i = \sqrt{\mu(2/r_1 - 1/a_1)}\,\boldsymbol{i}_n \times \boldsymbol{i}_{r1} \tag{16}$$

Similarly, let $\boldsymbol{v}_2$ be the transfer orbit velocity at P_2 and $\boldsymbol{v}_f$ be the target orbit velocity at P_2:

$$\Delta\boldsymbol{v}_2 = \boldsymbol{v}_f - \boldsymbol{v}_2 \tag{17}$$

$$\boldsymbol{v}_2 = \frac{1}{\eta}\sqrt{\frac{\mu}{a_m}}\left\{\left[2\lambda\frac{a_m}{r_2} - (\lambda + x\eta)\right]\boldsymbol{i}_{r2} + \left(\sqrt{\frac{r_1}{r_2}}\sin\frac{\Delta f}{2}\right)\boldsymbol{i}_n \times \boldsymbol{i}_{r2}\right\} \tag{18}$$

and

$$\boldsymbol{v}_f = \sqrt{\mu(2/r_2 - 1/a_2)}\,\boldsymbol{i}_n \times \boldsymbol{i}_{r2} \tag{19}$$

Then the total $\Delta\boldsymbol{v}_{\text{total}}$ is given as the sum of Eqs. (12) and (17) as

$$\Delta\boldsymbol{v}_{\text{total}} = \Delta\boldsymbol{v}_1 + \Delta\boldsymbol{v}_2 \tag{20}$$

where

$$(\Delta v_1)^2 = \left\{\frac{1}{\eta}\sqrt{\frac{\mu}{a_m}}\left[2\lambda\frac{a_m}{r_1} - (\lambda + x\eta)\right]\right\}^2 + \left[\frac{1}{\eta}\sqrt{\frac{\mu}{a_m}}\sqrt{\frac{r_2}{r_1}}\sin\frac{\Delta f}{2} - \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a_1}\right)}\right]^2 \tag{21}$$

and

$$(\Delta v_2)^2 = \left\{-\frac{1}{\eta}\sqrt{\frac{\mu}{a_m}}\left[2\lambda\frac{a_m}{r_2} - (\lambda + x\eta)\right]\right\}^2 + \left[-\frac{1}{\eta}\sqrt{\frac{\mu}{a_m}}\sqrt{\frac{r_1}{r_2}}\sin\frac{\Delta f}{2} + \sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a_2}\right)}\right]^2 \tag{22}$$

To minimize the fuel usage of a spacecraft, $\Delta\boldsymbol{v}_{\text{total}}$ must be minimized. Consequently, we minimize Eq. (20) with respect to x and find the minimum-fuel transfer.

Note that noncoplanar orbits may be considered in this manner to find the optimal-fuel and optimal-time elliptic transfer orbit. In the noncoplanar case Eq. (5) would remain the same; however, $\boldsymbol{i}_n$ in Eqs. (13) and (18) would change to $\boldsymbol{i}_{nt}$, $\boldsymbol{i}_n$ in Eq. (16) would change to $\boldsymbol{i}_{n1}$, and $\boldsymbol{i}_n$ in Eq. (19) would change to $\boldsymbol{i}_{n2}$. Here $\boldsymbol{i}_{nt}$ is a unit vector normal to the transfer orbital plane, $\boldsymbol{i}_{n1}$ is a unit vector normal to the initial orbital plane, and $\boldsymbol{i}_{n2}$ is a unit vector normal to the target orbital plane.

Minimum-Time Transfer Conditions

Derivation of the Transfer-Time Equation

The conditions to achieve two-impulse optimal-time transfer are derived. We show that the transfer time is a decreasing function of x. Thus minimum-time transfer does not have a definite solution, but it has a limiting solution in the sense that it is possible to find the limiting x such that the minimum-time transfer approaches this limit. The results are stated in this section, and the proofs are given in the Appendix.

Taking the derivative of Eq. (5) with respect to x, we obtain[10]

$$\sqrt{\frac{\mu}{a_m^3}}\frac{\mathrm{d}\Delta t}{\mathrm{d}x} = \frac{4}{3}\frac{\mathrm{d}F[3,1;5/2,(1-x)/2]}{\mathrm{d}x} - \frac{4}{3}\lambda^3\frac{\mathrm{d}F[3,1;5/2,(1-y)/2]}{\mathrm{d}x} \tag{23}$$

where $F[3, 1; 5/2; z]$ is the hypergeometric function whose continued fraction expansion is

$$F[3,1;5/2;z] = \cfrac{1}{1-\cfrac{\gamma_1 z}{1-\cfrac{\gamma_2 z}{1-\ddots}}} \tag{24}$$

and

$$\gamma_n = \left\{\begin{array}{ll} \dfrac{(n+2)(n+5)}{(2n+1)(2n+3)} & n \text{ odd} \\ \dfrac{n(n-3)}{(2n+1)(2n+3)} & n \text{ even}\end{array}\right\}$$

Furthermore, $F[3, 1; 5/2; z]$ can be rewritten as

$$F[3,1;5/2;z] = \frac{1}{1-\gamma_1 z G(z)} \tag{25}$$

where

$$G(z) = \cfrac{1}{1-\cfrac{\gamma_2 z}{1-\cfrac{\gamma_3 z}{1-\ddots}}}$$

and the derivative with respect to z is given as[10]

$$\frac{\mathrm{d}F[3,1;5/2;z]}{\mathrm{d}z} = \frac{3-9G(z)/5}{(1-z)[1-6zG(z)/5]} \tag{26}$$

To evaluate the first term on the right-hand side of Eq. (23), we let $z = (1-x)/2$. Then $\mathrm{d}z = (-\frac{1}{2})\mathrm{d}x$ and

$$\frac{\mathrm{d}F[3,1;5/2;z]}{\mathrm{d}z} = -2\frac{\mathrm{d}F[3,1;5/2;(1-x)/2]}{\mathrm{d}x} \tag{27}$$

Similarly, for the second term on the right-hand side of Eq. (23), we let $z = (1-y)/2$ to obtain

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{-2\lambda^2 x}{4\sqrt{1-\lambda^2(1-x^2)}}$$

and

$$\frac{\mathrm{d}F[3,1;5/2;z]}{\mathrm{d}z} = -\frac{2y}{\lambda^2 x}\frac{\mathrm{d}F[3,1;5/2;(1-y)/2]}{\mathrm{d}x} \tag{28}$$

Thus, we can evaluate Eq. (23) using Eqs. (27) and (28).

In the Appendix, using Eqs. (23), (27), and (28), we prove that the transfer-time equation is a monotonically decreasing function with respect to the variable x, i.e., $\mathrm{d}\Delta t/\mathrm{d}x < 0$. Consequently the largest value of x gives the fastest transfer time. For example, to find the fastest elliptic transfer orbit, we let x approach unity without actually reaching it ($x = 1$ corresponds to a parabolic orbit). Furthermore, this implies that the parabolic transfer orbits are faster than elliptic transfer orbits, and hyperbolic transfer orbits are faster than parabolic transfer orbits. Intuitively this states that if fuel is unlimited, the more delta velocity one expends the shorter the transfer time.

Necessary and Sufficient Conditions for Minimum-Fuel Transfer

The necessary and sufficient conditions for the minimum-fuel transfer in the case of a two-impulse elliptic transfer orbit are derived in this section. The initial and target orbits are assumed to be coplanar and coaxial elliptic orbits. We find the condition on the variable x (equivalent to the semimajor axis) such that the transfer orbit minimizes the fuel usage. The magnitudes of the delta velocities (Δv_1 and Δv_2) are given by Eqs. (21) and (22). By definition η is positive,[10] and it can be rewritten as

$$\eta = y - \lambda x = \sqrt{1-\lambda^2(1-x^2)} - \lambda x \tag{29}$$

Given Δf, the necessary and sufficient conditions for a minimum-fuel transfer orbit are given by

$$\frac{\mathrm{d}\Delta v_{\text{total}}}{\mathrm{d}x} = 0, \qquad \frac{\mathrm{d}^2\Delta v_{\text{total}}}{\mathrm{d}x^2} > 0$$

where

$$\frac{\mathrm{d}\Delta v_{\text{total}}}{\mathrm{d}x} = \frac{\mathrm{d}\Delta v_1}{\mathrm{d}x} + \frac{\mathrm{d}\Delta v_2}{\mathrm{d}x}$$

Necessary Condition

Thus, to find the necessary and sufficient conditions we take the first derivative of the delta velocities given in Eqs. (21) and (22) with respect to x. Also, we find the first derivative of Eq. (29) with respect to x. We obtain

$$\begin{aligned}\frac{\mathrm{d}\Delta v_1}{\mathrm{d}x} &= \frac{1}{\Delta v_1}\frac{\mu}{a_m}\left\{\frac{1}{\eta}\left[2\lambda\frac{a_m}{r_1} - (\lambda + x\eta)\right]\right\}\left(-\frac{1}{\eta^2}2\lambda\frac{a_m}{r_1}\frac{\mathrm{d}\eta}{\mathrm{d}x} + \frac{1}{\eta^2}\lambda\frac{\mathrm{d}\eta}{\mathrm{d}x} - 1\right) + \frac{1}{\Delta v_1}\left[\frac{1}{\eta}\sqrt{\frac{\mu}{a_m}}\sqrt{\frac{r_2}{r_1}}\sin\frac{\Delta f}{2}\right.\\ &\quad \left. - \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a_1}\right)}\right]\left(-\frac{1}{\eta^2}\sqrt{\frac{\mu}{a_m}}\sqrt{\frac{r_2}{r_1}}\sin\frac{\Delta f}{2}\frac{\mathrm{d}\eta}{\mathrm{d}x}\right)\end{aligned} \tag{30}$$

$$\begin{aligned}\frac{\mathrm{d}\Delta v_2}{\mathrm{d}x} &= \frac{1}{\Delta v_2}\frac{\mu}{a_m}\left\{-\frac{1}{\eta}\left[2\lambda\frac{a_m}{r_2} - (\lambda + x\eta)\right]\right\}\left(\frac{1}{\eta^2}2\lambda\frac{a_m}{r_2}\frac{\mathrm{d}\eta}{\mathrm{d}x} - \frac{1}{\eta^2}\lambda\frac{\mathrm{d}\eta}{\mathrm{d}x} + 1\right) + \frac{1}{\Delta v_2}\left[-\frac{1}{\eta}\sqrt{\frac{\mu}{a_m}}\sqrt{\frac{r_1}{r_2}}\sin\frac{\Delta f}{2}\right.\\ &\quad \left. + \sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a_2}\right)}\right]\left(\frac{1}{\eta^2}\sqrt{\frac{\mu}{a_m}}\sqrt{\frac{r_1}{r_2}}\sin\frac{\Delta f}{2}\frac{\mathrm{d}\eta}{\mathrm{d}x}\right)\end{aligned} \tag{31}$$

and

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} = \lambda^2\frac{x}{y} - \lambda \tag{32}$$

The first derivative of the total delta velocity

$$\frac{\mathrm{d}\Delta v_{\text{total}}}{\mathrm{d}x} = \frac{\mathrm{d}\Delta v_1}{\mathrm{d}x} + \frac{\mathrm{d}\Delta v_2}{\mathrm{d}x} \tag{33}$$

can be trivially obtained from Eqs. (30) and (31). Thus the necessary condition for a minimum requires

$$\frac{\mathrm{d}\Delta v_{\text{total}}}{\mathrm{d}x} = 0$$

Sufficient Condition

For the sufficient condition we find the second derivative of Eq. (32) with respect to x

$$\frac{d^2\eta}{dx^2} = \lambda^2 \frac{(y-x)(dy/dx)}{y^2} \tag{34}$$

Then we take the second derivative of the total delta velocity with respect to x. After another long manipulation we obtain

$$\begin{aligned}
\frac{d^2\Delta v_{\text{total}}}{dx^2} &= \frac{1}{\Delta v_1}\frac{\mu}{a_m}\left(12\frac{\lambda^2}{\eta^4}\frac{a_m^2}{r_1^2}\frac{d\eta}{dx} - \frac{4\lambda^2}{\eta^3}\frac{a_m^2}{r_1^2}\frac{d^2\eta}{dx^2} - 12\frac{\lambda^2}{\eta^4}\frac{a_m}{r_1}\frac{d\eta}{dx}\right.\\
&\left.+ \frac{4\lambda^2}{\eta^3}\frac{a_m}{r_1}\frac{d^2\eta}{dx^2}\right) + \frac{1}{\Delta v_1}\frac{\mu}{a_m}\left(-\frac{4\lambda x}{\eta^3}\frac{a_m}{r_1}\frac{d\eta}{dx} + \frac{2\lambda}{\eta^2}\frac{a_m}{r_1}\frac{d\eta}{dx}\right.\\
&\left.+ \frac{2\lambda x}{\eta^2}\frac{a_m}{r_1}\frac{d^2\eta}{dx^2} + \frac{3\lambda^2}{\eta^4}\frac{d\eta}{dx}\right) + \frac{1}{\Delta v_1}\frac{\mu}{a_m}\left(-\frac{\lambda^2}{\eta^3}\frac{d^2\eta}{dx^2} + \frac{2\lambda x}{\eta^3}\frac{d\eta}{dx}\right.\\
&\left.- \frac{\lambda}{\eta^2}\frac{d\eta}{dx} - \frac{\lambda x}{\eta^2}\frac{d^2\eta}{dx^2} + \frac{2\lambda a_m}{\eta^2 r_1} - \frac{\lambda}{\eta^2} + 1\right) + \frac{1}{\Delta v_1}\left(\frac{3}{\eta^4}\frac{d\eta}{dx}\right.\\
&\left.- \frac{1}{\eta^3}\frac{d^2\eta}{dx^2}\right)\left[\frac{\mu}{a_m}\frac{r_2}{r_1}\left(\sin^2\frac{\Delta f}{2}\right)\right] + \frac{1}{\Delta v_1}\left(-\frac{2}{\eta^3}\frac{d\eta}{dx}\right.\\
&\left.+ \frac{1}{\eta^2}\frac{d^2\eta}{dx^2}\right)\left[\sqrt{\frac{\mu}{a_m}\frac{r_2}{r_1}}\left(\sin\frac{\Delta f}{2}\right)\sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a_1}\right)}\right]\\
&+ \frac{1}{\Delta v_2}\frac{\mu}{a_m}\left(12\frac{\lambda^2}{\eta^4}\frac{a_m^2}{r_2^2}\frac{d\eta}{dx} - \frac{4\lambda^2}{\eta^3}\frac{a_m^2}{r_2^2}\frac{d^2\eta}{dx^2} - 12\frac{\lambda^2}{\eta^4}\frac{a_m}{r_2}\frac{d\eta}{dx}\right.\\
&\left.+ \frac{4\lambda^2}{\eta^3}\frac{a_m}{r_2}\frac{d^2\eta}{dx^2}\right) + \frac{1}{\Delta v_2}\frac{\mu}{a_m}\left(-\frac{4\lambda x}{\eta^3}\frac{a_m}{r_2}\frac{d\eta}{dx} + \frac{2\lambda}{\eta^2}\frac{a_m}{r_2}\frac{d\eta}{dx}\right.\\
&\left.+ \frac{2\lambda x}{\eta^2}\frac{a_m}{r_2}\frac{d^2\eta}{dx^2} + \frac{3\lambda^2}{\eta^4}\frac{d\eta}{dx}\right) + \frac{1}{\Delta v_2}\frac{\mu}{a_m}\left(-\frac{\lambda^2}{\eta^3}\frac{d^2\eta}{dx^2}\right.\\
&\left.+ \frac{2\lambda x}{\eta^3}\frac{d\eta}{dx} - \frac{\lambda}{\eta^2}\frac{d\eta}{dx} - \frac{\lambda x}{\eta^2}\frac{d^2\eta}{dx^2} + \frac{2\lambda a_m}{\eta^2 r_2} - \frac{\lambda}{\eta^2} + 1\right)\\
&+ \frac{1}{\Delta v_1}\left(\frac{3}{\eta^4}\frac{d\eta}{dx} - \frac{1}{\eta^3}\frac{d^2\eta}{dx^2}\right)\left[\frac{\mu}{a_m}\frac{r_1}{r_2}\left(\sin^2\frac{\Delta f}{2}\right)\right]\\
&+ \frac{1}{\Delta v_2}\left(-\frac{2}{\eta^3}\frac{d\eta}{dx} + \frac{1}{\eta^2}\frac{d^2\eta}{dx^2}\right)\\
&\times\left[\sqrt{\frac{\mu}{a_m}\frac{r_1}{r_2}}\left(\sin\frac{\Delta f}{2}\right)\sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a_2}\right)}\right]
\end{aligned} \tag{35}$$

This second derivative must be greater than zero for Δv_{total} to be minimum. Thus we have the sufficient condition for a local minimum:

$$\frac{d^2\Delta v_{\text{total}}}{dx^2} > 0 \tag{36}$$

Equating Eq. (33) to zero gives the necessary condition, and Eq. (36) gives the sufficient condition to achieve minimum-fuel transfer for any two points in space (i.e., Δf can be anywhere between 0 and 360 deg).

Coplanar Hohmann Transfer from Lambert's Theorem

Necessary and Sufficient Conditions for a Minimum

To verify the results of the preceding section, we assume elliptic coplanar coaxial orbits and derive the necessary and sufficient conditions for the Hohmann transfer case. By Hohmann transfer we mean that tangential impulses are applied at opposing apses. Consequently, the initial and target points must be on the line of apsides for Hohmann transfer. We show that for the Hohmann transfer x is equal to zero. In the Hohmann transfer we have $\Delta f = \pi$ and $c = r_1 + r_2$. Thus, from Eqs. (1) and (2) we get $\lambda = 0$. Moreover, because $\lambda = 0$, from Eq. (14) we obtain $\eta = 1$ (η is positive by definition). Substituting these values into Eq. (33) gives the following result:

$$\frac{d\Delta v_{\text{total}}}{dx} = \frac{\mu}{a_m}x\left(\frac{1}{\Delta v_1} + \frac{1}{\Delta v_2}\right) \tag{37}$$

Thus $d\Delta v_{\text{total}}/dx$ is equal to zero when x is equal to zero. Consequently, for the case of a Hohmann transfer ($\Delta f = \pi$) we obtain the optimal fuel transfer when x is equal to zero. This shows that condition (37) is a special case of the general condition (33).

To find the sufficient condition, we evaluate Eq. (35) for the Hohmann transfer. Then we obtain

$$\frac{d^2\Delta v_{\text{total}}}{dx^2} = \frac{\mu}{a_m}\left(\frac{1}{\Delta v_1} + \frac{1}{\Delta v_2}\right) > 0 \tag{38}$$

Thus, this gives the sufficient condition for a minimum. Moreover, this shows that for the elliptic-to-elliptic, coplanar, and coaxial orbits if $\Delta f = \pi$ then we have minimum-fuel transfer when $x = 0$.

Note that $x = 0$ corresponds to the minimum energy transfer[10]; for Hohmann transfer the minimum-energy transfer corresponds to the minimum-fuel transfer, but as will be seen in the simulation section, when $\Delta f \neq \pi$ this is not the case.

Derivation of Transfer-Time and Delta-Velocity Equations

We now show that the well-known Hohmann transfer results are obtained from the Lagrange form of the transfer time and the velocity equations. For the Hohmann transfer $c = s = r_1 + r_2$ and $a = (r_1 + r_2)/2$. Thus, from Eqs. (10) and (11) we find that $\alpha = \pi$ and $\beta = 0$. Thus, using the fact that $\lambda = 0$ and $a_m = (r_1 + r_2)/2$ in Eq. (5) we obtain

$$\Delta t = \pi\sqrt{\frac{(r_1 + r_2)^3}{8\mu}}$$

In summary, from the Lagrange form of the transfer time Eq. (5), we obtain the Hohmann transfer-time equation

$$\Delta t = \pi\sqrt{\frac{(r_1 + r_2)^3}{8\mu}} \tag{39}$$

when Δf is equal to 180 deg. We note that this transfer-time equation is valid for elliptic-to-elliptic coaxial orbits as well as the circle-to-circle orbits.

Now, we derive $\Delta \boldsymbol{v}_{\text{total}}$ for the Hohmann transfer using Eq. (20) to verify that we recover the Hohmann transfer $\Delta \boldsymbol{v}_{\text{total}}$. Once again, for the Hohmann transfer we have $c = s = r_1 + r_2$, $\alpha = \pi$, $\beta = 0$, $\lambda = 0$, and $a_m = (r_1 + r_2)/2 = a$. The last equation implies $x = 0$. Because $\eta^2 = (1 - \lambda^2) + 4\lambda\sin^2(\psi/2)$ and $\psi = (\alpha - \beta)/2$ for an elliptic orbit, we obtain $\eta = 1$ and $\psi = \pi/2$. Substituting these values into Eq. (13), we obtain

$$\boldsymbol{v}_1 = \sqrt{\frac{\mu}{a}}\left[\left(\sqrt{\frac{r_2}{r_1}}\sin\frac{\pi}{2}\right)\boldsymbol{i}_n \times \boldsymbol{i}_{r1}\right] \tag{40}$$

Using Eq. (12), we get

$$\Delta\boldsymbol{v}_1 = \left[\sqrt{\frac{\mu}{a}}\sqrt{\frac{r_2}{r_1}} - \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a_1}\right)}\right]\boldsymbol{i}_n \times \boldsymbol{i}_{r1} \tag{41}$$

By similar analysis we obtain

$$\Delta\boldsymbol{v}_2 = \left[\sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a_2}\right)} - \sqrt{\frac{\mu}{a}}\sqrt{\frac{r_1}{r_2}}\right]\boldsymbol{i}_n \times \boldsymbol{i}_{r2} \tag{42}$$

Using the relations $r_1 = a(1 - e_1)$ and $r_2 = a(1 + e_2)$, we obtain

$$\Delta\boldsymbol{v}_1 = \sqrt{\frac{\mu}{r_1}}\left[\sqrt{\frac{2(r_2/r_1)}{1 + (r_2/r_1)}} - \sqrt{1 + e_1}\right]\boldsymbol{i}_n \times \boldsymbol{i}_{r1} \tag{43}$$

and

$$\Delta \boldsymbol{v}_2 = \sqrt{\frac{\mu}{r_2}}\left[\sqrt{1+e_2} - \sqrt{\frac{2}{1+(r_2/r_1)}}\right]\boldsymbol{i}_n \times \boldsymbol{i}_{r2} \tag{44}$$

The magnitude of the sum of the preceding two equations gives the desired result.

As for the special case of circle-to-circle transfer, we have $a_1 = r_1$ and $a_2 = r_2$. We get

$$\Delta \boldsymbol{v}_1 = \left(\sqrt{\frac{\mu}{a}}\sqrt{\frac{r_2}{r_1}} - \sqrt{\frac{\mu}{r_1}}\right)\boldsymbol{i}_n \times \boldsymbol{i}_{r1}$$

$$= \sqrt{\mu}\left(\sqrt{\frac{2}{r_1} - \frac{1}{a}} - \sqrt{\frac{1}{r_1}}\right)\boldsymbol{i}_n \times \boldsymbol{i}_{r1} \tag{45}$$

By similar analysis we also obtain

$$\Delta \boldsymbol{v}_2 = \left(\sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{\mu}{a}}\sqrt{\frac{r_1}{r_2}}\right)\boldsymbol{i}_n \times \boldsymbol{i}_{r2}$$

$$= \sqrt{\mu}\left(\sqrt{\frac{1}{r_2}} - \sqrt{\frac{2}{r_2} - \frac{1}{a}}\right)\boldsymbol{i}_n \times \boldsymbol{i}_{r2} \tag{46}$$

From the $\Delta \boldsymbol{v}_{\text{total}}$ Eq. (20), we recover the well-known Hohmann total delta velocity $\Delta \boldsymbol{v}_{\text{total}}$

$$\Delta \boldsymbol{v}_{\text{total}} = \sqrt{\mu}\left[\left|\left(\frac{2}{r_1} - \frac{1}{a}\right)^{\frac{1}{2}} - \left(\frac{1}{r_1}\right)^{\frac{1}{2}}\right| + \left|\left(\frac{2}{r_2} - \frac{1}{a}\right)^{\frac{1}{2}} - \left(\frac{1}{r_2}\right)^{\frac{1}{2}}\right|\right] \tag{47}$$

when Δf is equal to 180 deg.

Note that the Hohmann transfer is a special case for Δf equal to 180 deg. Thus for $\Delta f \neq 180$ deg the Hohmann transfer cannot be used; however, we can still obtain the minimum-fuel or minimum-time transfer orbit using the method described in the preceding section.

Optimal Coplanar Orbit Maneuver Algorithm

Here, a method to perform the optimal-fuel and optimal-time maneuver using Lambert's theorem is presented. The problem is to find the transfer orbit's semimajor axis such that the transfer time and the fuel used is minimum. From the results of the preceding section, we note that the Hohmann transfer is the optimal solution when $\Delta f = \pi$. However, if we wish to put the spacecraft at a particular point in the target orbit where $\Delta f \neq \pi$, then the Hohmann transfer cannot be performed, and the minimum-fuel transfer is obtained from Eqs. (30–33). It is important to note that Eqs. (30–33) is valid even if the initial and target points are not on the line of apsides. This type of transfer might be necessary in the case of interception and rendezvous.

Now, we use the results of the preceding sections to obtain the optimal-fuel or optimal-time-transfer orbit. The optimal algorithm is given as follows:

1) Given initial orbit radius r_1, target orbit radius r_2, and the difference in the true anomaly (transfer angle) Δf, we vary x to find the transfer time Δt and total delta velocity $\Delta \boldsymbol{v}_{\text{total}}$.

2) Find the cord length c from the equation

$$c^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\Delta f \tag{48}$$

3) Find the semiperimeter of the space triangle s from Eq. (1).

4) Find a_m from Eq. (4).

5) Find a constant λ from Eq. (2).

Note that the cord length c can also be expressed in terms of this variable as

$$c = (r_1 + r_2)[(1-\lambda^2)/(1+\lambda^2)] \tag{49}$$

Thus, λ is related to the cord length.

6) Find α from Eq. (6).

7) Find β from Eq. (7).

8) Find the transfer time from the Lagrange form of the transfer time Eq. (5).

9) Find the total delta velocity from Eqs. (20–22).

Equation (5) can be used to find the transfer orbit's semimajor axis such that the transfer time is minimized. We can plot Δt (which is related to the transfer orbit semimajor axis) vs x and find x such that Δt is minimum. Actually it is known from the previous section that Δt is a decreasing function of x. Thus, e.g., for an elliptic transfer orbit the smallest Δt is achieved by letting x approach unity without actually reaching it ($x = 1$ corresponds to a parabolic orbit). Thus, just to perform the elliptic minimum-time orbit transfer we let x be very close to unity (for elliptic transfer orbit $x < 1$).

There is another factor to take into consideration when we perform an optimal maneuver. We would like to minimize fuel usage while transferring to the target orbit. We can plot $\Delta \boldsymbol{v}_{\text{total}}$ (which is related to the transfer orbit semimajor axis) vs x using Eqs. (20–22) and find x such that $\Delta \boldsymbol{v}_{\text{total}}$ is minimum to find the optimal-fuel transfer orbit. Thus, there is a tradeoff between the time the spacecraft takes to get to the target orbit and the fuel necessary to get to the target orbit.

Numerical Simulation Results

The numerical simulation is performed in this section using the optimal maneuver algorithm developed in the preceding section. Here we let $r_1 = 6700$ km, $r_2 = 6710$ km, $a_1 = 6800$ km, and $a_2 = 6900$ km. The first example is when $\Delta f = \pi$ (Hohmann transfer), and the second example is when $\Delta f = \pi/10$.

Case 1 ($\Delta f = \pi$: Hohmann Transfer)

We plot $|\Delta \boldsymbol{v}_{\text{total}}|$ vs x for a Hohmann transfer in Fig. 2. Note that minimum $|\Delta \boldsymbol{v}_{\text{total}}|$, which has the value 0.1619 km/s, corresponds to $x = 0$, which represents the Hohmann transfer. Using Eqs. (33) and (35), we have verified that $\mathrm{d}\Delta \boldsymbol{v}_{\text{total}}/\mathrm{d}x = 0$ and $\mathrm{d}^2\Delta \boldsymbol{v}_{\text{total}}/\mathrm{d}x^2 > 0$ when $x = 0$. Thus $x = 0$ corresponds to the minimum-fuel solution.

Figure 3 shows transfer time vs x, which is a monotonically decreasing function. When x is equal to zero, the transfer time is 2732.0 s. It is possible to obtain a transfer orbit that will give a smaller transfer time, but the fuel used will increase. Thus there is tradeoff between the fuel used and minimum time. Although it is not shown on the plot, the parabolic transfer orbit has smaller transfer time than an elliptic transfer orbit, and the hyperbolic transfer orbit has a smaller transfer time than a parabolic transfer orbit.

To see the relationship between the transfer orbit's semimajor axis and x, see Fig. 4. Note that even though there is only one a for the given x, there are two x for the given a. The semimajor axis of the transfer orbit corresponding to x equal to zero is 6721.8 km.

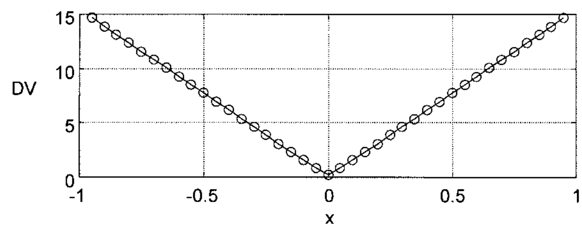

Fig. 2 $|\Delta v_{\text{total}}|$ **vs** x **(**$\Delta f = \pi$**).**

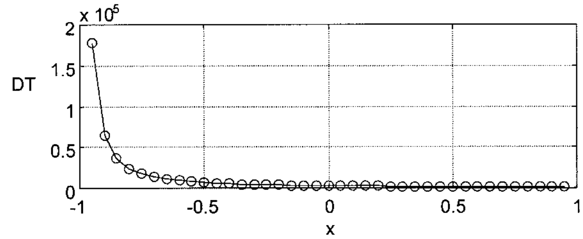

Fig. 3 Transfer time vs x **(**$\Delta f = \pi$**).**

Table 1 Comparison of case 1 and case 2

Case	x	Δv_{total}, km/s	Δt, s	a, km
Case 1: $\Delta f = \pi$	0.00	0.1619	2732.0	6721.8
Case 2: $\Delta f = \pi/10$	0.65	0.1703	273.03	6713.3
Difference	0.65	0.0084	−2459.0	−8.5000

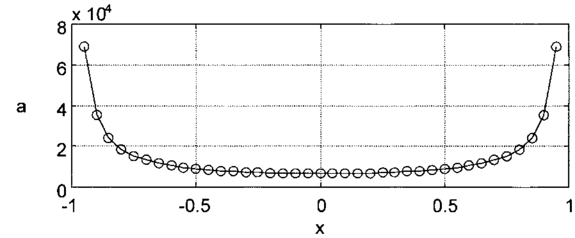

Fig. 4 Transfer orbit semimajor axis vs x.

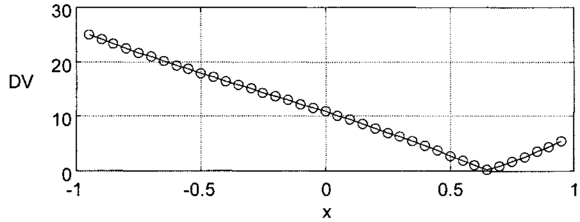

Fig. 5 $|\Delta v_{\text{total}}|$ vs x ($\Delta f = \pi/10$).

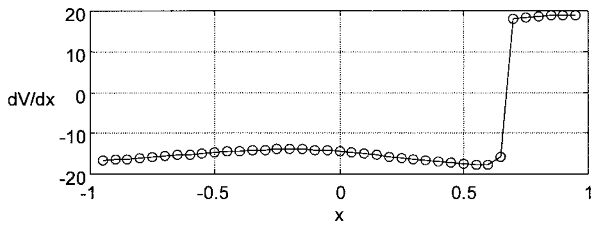

Fig. 6 $d\Delta v_{\text{total}}/dx$ vs x ($\Delta f = \pi/10$).

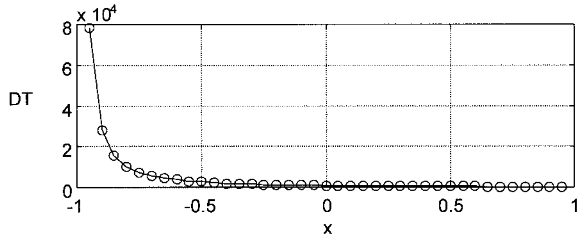

Fig. 7 Transfer time vs x ($\Delta f = \pi/10$).

Case 2 ($\Delta f = \pi/10$)

In the case of rendezvous, we may require that $\Delta f \neq \pi$. Thus the question is what is the optimal-fuel or optimal-time transfer orbit when $\Delta f \neq \pi$. Here we assume that the two points are separated by $\Delta f = \pi/10$. Figure 5 shows $|\Delta v_{\text{total}}|$ vs x. Note that the minimum $|\Delta v_{\text{total}}|$ is achieved when x is approximately 0.65. Note also that the minimum energy orbit ($x = 0$) does not correspond to the minimum-fuel orbit. Using Eqs. (33) and (35), we have verified that $d\Delta v_{\text{total}}/dx = 0$ and $d^2\Delta v_{\text{total}}/dx^2 > 0$ when $x \approx 0.649$. Thus $x \approx 0.649$ corresponds to the minimum-fuel solution. See the $d\Delta v_{\text{total}}/dx$ vs x graph in Fig. 6. We note that $d\Delta v_{\text{total}}/dx$ is negative when $x < 0.649$, and it becomes positive when $x > 0.649$, which implies that $x \approx 0.649$ is a minimum.

Figure 7 shows the transfer time vs x. Note that it is a monotonically decreasing function as before. Thus for this problem the optimal-fuel and time-transfer orbit occurs when $x = 0.65$, which corresponds to $a = 6.7133 \times 10^3$ km. For $x = 0.65$, $\Delta v_{\text{total}} = 0.1703$ km/s and $\Delta t = 273.03$ s.

To compare these values with the Hohmann transfer case, see Table 1. Note that Δv_{total} is smaller for the Hohmann transfer (case 1) as expected, but the transfer time is much larger.

Conclusions

The transfer-time and delta-velocity equations are stated using Lambert's theorem. Using these equations minimum-time and minimum-fuel conditions in terms of the semimajor axis of the elliptic transfer orbit are derived. Moreover, the minimum-time transfer orbit approaches a limiting value. Also, when the transfer angle is 180 deg the optimal-fuel transfer is given by the well-known Hohmann transfer. An algorithm with the fuel and time tradeoff is presented. This procedure can be used to find the optimal-time and optimal-fuel transfer orbit in the case of a rendezvous.

Appendix: Proof That Transfer Time Is a Decreasing Function of x

In this Appendix we prove that the transfer time Eq. (23) is a monotonically decreasing function with respect to x. We show that the first term on the right-hand side of Eq. (23) is negative. Then we show that the second term's magnitude is smaller than the first term, thus showing that $d\Delta t/dx < 0$.

Now, we are ready to show that $dF[3, 1; 5/2, (1-x)/2]/dx$ is less than zero. First we find $F[3, 1; 5/2; z]$ using the algorithm given by Battin,[10] which is repeated here for the sake of completeness.

Initialize $\delta_1 = u_1 = \Sigma_1 = 1$, and calculate

$$\delta_{n+1} = \frac{1}{1 - \gamma_n z \delta_n}, \qquad u_{n+1} = u_n(\delta_{n+1} - 1)$$
$$\Sigma_{n+1} = \Sigma_n + u_{n+1} \tag{A1}$$

where γ_n is given in Eq. (24). Then for $z < 1$, $F[3, 1; 5/2; z] = \lim_{n\to\infty} \Sigma_n$. Second, we find $G(z)$ from Eq. (25). Third, we find dF/dz from Eq. (26). Finally, we find $dF[3, 1; 5/2; (1-x)/2]/dx$ using Eq. (27). The result is shown in Fig. A1, Although it is not clearly shown in Fig. A1 all of the values are less than zero for x between −1 and +1 (see Table A1).

Thus, this shows that for $-1 < x < 1$

$$\frac{4}{3}\frac{dF[3, 1; 5/2; (1-x)/2]}{dx} < 0 \tag{A2}$$

We compare the magnitude of the first and second terms on the right-hand side of Eq. (23). For the elliptic transfer orbit $x \in (-1, 1)$ and $\lambda \in [-1, 1]$. Using this information, we plot $\lambda^5 x/y$ vs x in Fig. A2 for several values of λ ranging between +1 and −1. We note that

$$|\lambda^5 x/y| \leq 1 \tag{A3}$$

Substitute Eq. (8) into $\lambda^5 x/y$, and obtain

$$\frac{\lambda^5 x}{y} = \frac{\lambda^5 x}{\sqrt{1 - \lambda^2(1 - x^2)}} \tag{A4}$$

Table A1 Hypergeometric function values

x	$\frac{dF[3, 1; 5/2; (1-x)/2]}{dx}$
−1	$-\infty$
−0.99	-2.639×10^5
⋮	⋮
0.99	-6.069×10^{-1}
1	Undefined

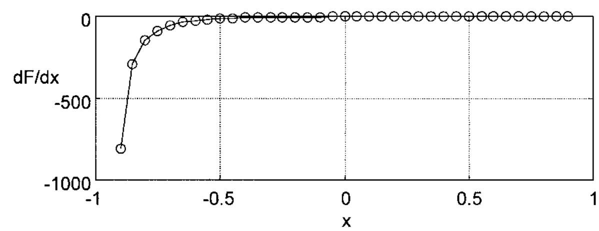

Fig. A1 $dF[3, 1; 5/2; (1-x)/2]/dx$ vs x.

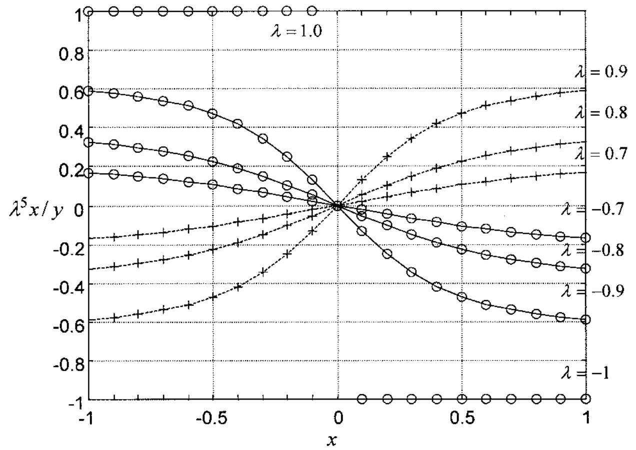

Fig. A2 $\lambda^5 x/y$ vs x for various values of λ.

Then we note that $\lambda^5 x/y = 1$ when $\lambda = 1$ and $\lambda^5 x/y = -1$ when $\lambda = -1$ for all x. Using Inequality (A3), we obtain

$$\left|\frac{4}{3}\left(-\frac{1}{2}\right)\frac{\mathrm{d}F}{\mathrm{d}z}\right| \geq \left|\frac{4}{3}\left(-\frac{1}{2}\right)\left(\frac{\lambda^5 x}{y}\right)\frac{\mathrm{d}F}{\mathrm{d}z}\right| \tag{A5}$$

Then we use Eqs. (27) and (28) in the preceding inequality to obtain

$$\left|\frac{4}{3}\frac{\mathrm{d}F[3,\,1;\,5/2;\,(1-x)/2]}{\mathrm{d}x}\right| \geq \left|\frac{4}{3}\lambda^3\frac{\mathrm{d}F[3,\,1;\,5/2;\,(1-y)/2]}{\mathrm{d}x}\right| \tag{A6}$$

Because of Inequalities (A2) and (A6), we have the following:

$$\frac{4}{3}\frac{\mathrm{d}F[3,\,1;\,5/2;\,(1-x)/2]}{\mathrm{d}x} - \frac{4}{3}\lambda^3\frac{\mathrm{d}F[3,\,1;\,5/2;\,(1-y)/2]}{\mathrm{d}x} \leq 0 \tag{A7}$$

Finally, because of Eqs. (23) and (A7), we have the desired result:

$$\frac{\mathrm{d}\Delta t}{\mathrm{d}x} \leq 0 \tag{A8}$$

Thus, Δt is a decreasing function with respect to x.

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